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Graded contractions of the affine Lie algebra $A_1^{(1)}$, its representations and tensor products, and an application to the branching rule $A_1^{(1)} \supset A_1^{(1)}$

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Abstract. We describe all contractions of the affine Kac–Moody algebra $A_1^{(1)}$ which preserve various gradings by the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as well as a grading by $\mathbb{Z}_2 \times \mathbb{Z}_2$, together with the contractions of simultaneously graded representations of $A_1^{(1)}$ and their tensor products. An application is given to the branching rules for $A_1^{(1)} \supset A_1^{(1)}$ for a number of distinct embeddings, each associated with a particular grading.

1. Introduction

Deformations of a Lie algebra by means of singular transformations to its tensor of structure constants is a standard procedure in mathematics [1–3] and in physics [4, 5], although physicists often restrict themselves to much less general situations in practice. Such deformations of a Lie algebra are known as contractions. A major handicap of this approach is that it does not extend the deformation procedure in any obvious way to an important part of Lie theory, namely the theory of representations of Lie algebras. Indeed, there are apparently no deformations of representations to be found in the mathematics literature, although some appear in the physics literature, for example in the work of Gromov [6, 7] who has explicitly developed a deformation of representations of unitary Cayley–Klein algebras based on contractions. The deformations of Lie algebras preserving a grading by a finite Abelian group G as introduced in [8, 9] and developed further in [10–12] not only extend naturally to all representations but allows one to simultaneously study deformations of all those Lie algebras, both finite and infinite, which admit a grading by G.

An investigation of contractions of an infinite-dimensional Lie algebra is undertaken here apparently for the first time, although the applicability of the method to this case is implicit in [9]. We study contractions of the affine Kac–Moody algebra $A_1^{(1)}$ and its representations which preserve $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings. The aim in this article is twofold: first, to use $A_1^{(1)}$ as a laboratory to illustrate and explore various aspects of graded contractions as applied not only to algebras but also to their representations and even tensor products of their representations; and second, to explore specific properties of the structure and representation theory of a Kac–Moody algebra. Although from the standpoint of the graded contractions the computations here are not much different from those in [8–12], the result of the contractions give rise to some new phenomena not encountered in the finite case. Thus for example a contraction of $A_1^{(1)}$ may turn out to be an algebra which is a semidirect product of $A_1^{(1)}$ with an infinite-dimensional Abelian ideal—an 'inhomogeneous $A_1^{(1)}$'. Moreover, we will illustrate the phenomenon, not possible for finite-dimensional Lie algebras, whereby a Kac–Moody algebra may be maximally embedded in itself.

In section 2 we briefly describe how graded contractions are determined. In particular we recall the three sets of equations for the contractions, respectively, of Lie algebras, their representations and tensor products of their representations. Relevant $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_2\times\mathbb{Z}_2$-gradings of $A_1^{(1)}$ are described explicitly in section 3. The corresponding contractions of the Lie algebra are determined in section 4. For every contraction of the Lie algebra there is a menu of possible contractions of the representations. In sections 5 and 6 we find, respectively, the corresponding contractions of representations of the Lie algebra and their tensor products. Note that this analysis is done without fixing a specific representations, it applies to all of them. The gradings of particular highest-weight representations are described in section 7. An application of the graded contraction to the computation of branching rules for various embeddings of $A_1^{(1)}$ in $A_1^{(1)}$ is made in section 8. These embeddings and branching rules are different from those previously identified [13].

2. Graded contractions

In this section we briefly overview the method of graded contractions following [8–12]. It is convenient to introduce some notation appropriate to the case of a general Lie algebra L before embarking later on a study of $A_1^{(1)}$.

Let L be a Lie algebra, finite or infinite dimensional, graded by a finite Abelian group, G, so that

$$L = \bigoplus_{j\in G} L_j \qquad \text{with} \quad [L_j, L_k] \subseteq L_{j+k} \quad \text{for } j, k \in G \tag{2.1}$$

where $[L_j, L_k]$ is the linear space generated by the products in L of every element of L_j with every element of L_k. The product in L has been indicated by $[\cdot,\cdot]$, and that in G by $+$.

The Lie algebra L is said to have G-graded structure κ, where the symmetric matrix $\kappa = (\kappa_{jk})$ is defined by

$$\kappa_{jk} = \begin{cases} 0 & \text{if} \quad [L_j, L_k] = 0 \\ 1 & \text{if} \quad [L_j, L_k] \neq 0 . \end{cases} \tag{2.2}$$

The G-graded contraction, L^ϵ, of L is the Lie algebra isomorphic to L as a vector space and sharing the same graded decomposition (2.1), so that $L_j^\epsilon := L_j$, but whose product is defined by

$$[L_j^\epsilon, L_k^\epsilon]_\epsilon := \epsilon_{jk}[L_j, L_k] \subseteq \epsilon_{jk} L_{j+k}^\epsilon . \tag{2.3}$$

From (2.2), the antisymmetry of $[\cdot,\cdot]$ and the Jacobi identity it follows that the contraction matrix ϵ satisfies the constraints

$$\epsilon_{jk} = * \qquad \text{if} \quad \kappa_{jk} = 0 \tag{2.4a}$$

$$\epsilon_{jk} = \epsilon_{kj} \tag{2.4b}$$

and

$$\epsilon_{jk}\epsilon_{m,j+k} = \epsilon_{km}\epsilon_{j,k+m} = \epsilon_{mj}\epsilon_{k,m+j} . \tag{2.4c}$$

In (2.4a) the value $*$ indicates that ϵ_{jk} is arbitrary. Any equation containing $*$ which is obtained from (2.4c) is to be dropped since the corresponding constraint is not required.

A G-graded decomposition (2.1) of L leads to a G-grading of any L-module V:

$$V = \bigoplus_{m\in G} V_m \qquad \text{where} \quad L_j V_m \subseteq V_{j+m} \tag{2.5}$$

and

$$[L_j, L_k]V_m = L_j L_k V_m - L_k L_j V_m \subseteq V_{j+k+m}\,. \tag{2.6}$$

In the generic case $L_j V_m \neq 0$, but in general let

$$\lambda_{jm} = \begin{cases} 0 & \text{if } L_j V_m = 0 \\ 1 & \text{if } L_j V_m \neq 0\,. \end{cases} \tag{2.7}$$

The L-module V may be converted into an L^ϵ-module $V^{\epsilon,\psi}$, with $V_m^{\epsilon,\psi} := V_m$, through the introduction of a matrix ψ of additional contraction parameters ψ_{jm}. In terms of these parameters:

$$L_j^\epsilon V_m^{\epsilon,\psi} := \psi_{jm} L_j V_m \subseteq \psi_{jm} V_{j+m}^{\epsilon,\psi} \tag{2.8}$$

and

$$[L_j^\epsilon, L_k^\epsilon]_\epsilon V_m^{\epsilon,\psi} := \psi_{km}\psi_{j,k+m} L_j L_k V_m - \psi_{jm}\psi_{k,j+m} L_k L_j V_m\,. \tag{2.9}$$

However, (2.3) implies

$$[L_j^\epsilon, L_k^\epsilon]_\epsilon V_m^{\epsilon,\psi} \subseteq \epsilon_{jk}\psi_{j+k,m} L_{j+k}^\epsilon V_m^{\epsilon,\psi}\,. \tag{2.10}$$

For consistency it is necessary that

$$\psi_{jm} = * \qquad \text{if} \quad \lambda_{jm} = 0 \tag{2.11a}$$

and

$$\psi_{km}\psi_{j,k+m} = \psi_{jm}\psi_{k,m+j} = \epsilon_{jk}\psi_{j+k,m} \qquad \text{if} \quad \kappa_{jk} \neq 0\,. \tag{2.11b}$$

Any consistency condition obtained from (2.11b) which involves $*$, arising from either (2.4a) or (2.11a), is to be dropped.

Finally if V and W are G-graded L-modules, then the tensor product module $V \otimes W$ is correspondingly graded:

$$V \otimes W = \bigoplus_{p\in G} (V \otimes W)_p \qquad \text{with} \quad (V \otimes W)_p := \bigoplus_{m+n=p} V_m \otimes W_n \tag{2.12}$$

so that

$$V_m \otimes W_n \subseteq (V \otimes W)_{m+n} \tag{2.13}$$

and

$$L_j(V\otimes W)_p=\sum_{m+n=p}\left\{\left((L_jV_m)\otimes W_n\right)+\left(V_m\otimes(L_jW_n)\right)\right\}\subseteq(V\otimes W)_{j+p}\,. \tag{2.14}$$

Carrying out simultaneous ψ-contractions of V and W with respect to ϵ and introducing yet another matrix, τ, of contraction parameters τ_{mn}, results in the conversion of the G-graded tensor product module into an L^ϵ-module $(V\otimes W)^{\epsilon,\psi,\tau}$ with

$$(V\otimes W)_p^{\epsilon,\psi,\tau}:=\sum_{m+n=p}\tau_{mn}V_m^{\epsilon\psi}\otimes W_n^{\epsilon\psi}\,. \tag{2.15}$$

The action of L^ϵ is defined equally well by

$$L_j^\epsilon(V\otimes W)_p^{\epsilon,\psi,\tau}:=\sum_{m+n=p}\tau_{mn}L_j^\epsilon\left(V_m^{\epsilon,\psi}\otimes W_n^{\epsilon,\psi}\right) \tag{2.16}$$

or

$$L_j^\epsilon(V\otimes W)_p^{\epsilon,\psi,\tau}:=\sum_{m+n=p}\left\{\tau_{j+m,n}(L_j^\epsilon V_m^{\epsilon,\psi})\otimes W_n^{\epsilon,\psi}+\tau_{m,j+n}V_m^{\epsilon,\psi}\otimes(L_j^\epsilon W_n^{\epsilon,\psi})\right\}. \tag{2.17}$$

Noting that (2.15) implies

$$V_m^{\epsilon\psi}\otimes W_n^{\epsilon\psi}\subseteq(V\otimes W)_p^{\epsilon,\psi,\tau}\qquad\text{if}\quad\tau_{mn}\neq0 \tag{2.18}$$

it follows from the use of (2.8) in (2.16) and (2.17) that

$$\psi_{j,m+n}\tau_{mn}=\psi_{jm}\tau_{j+m,n}=\psi_{jn}\tau_{m,j+n}\,. \tag{2.19}$$

In what follows the case $L=A_1^{(1)}$ will be treated in detail for the three grading groups $G=\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2\times\mathbb{Z}_2$. For each of these grading groups a number of different gradings may be identified. For each grading all possible ϵ, ψ and τ may be found by successively solving (2.4), (2.11) and (2.19).

3. Gradings of $A_1^{(1)}$

The three-dimensional simple Lie algebra A_1 has basis h, e and f with

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad e=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad f=\begin{pmatrix}0&0\\1&0\end{pmatrix} \tag{3.1}$$

with products defined by commutators so that:

$$[h,h]=[e,e]=[f,f]=0\qquad[h,e]=2e\qquad[h,f]=-2f\qquad[e,f]=h\,. \tag{3.2}$$

The corresponding infinite-dimensional affine Kac–Moody algebra $A_1^{(1)}$ has basis h_k, e_k, f_k and $\not{c}$ with $k\in\mathbb{Z}$, where

$$h_k=\begin{pmatrix}t^k&0\\0&-t^k\end{pmatrix}\qquad e_k=\begin{pmatrix}0&t^k\\0&0\end{pmatrix}\qquad f_k=\begin{pmatrix}0&0\\t^k&0\end{pmatrix}\qquad\not{c}=c\begin{pmatrix}1&0\\0&1\end{pmatrix} \tag{3.3}$$

where c is an arbitrary complex parameter. Products are defined by a slight modification of commutators necessitated by the fact that (3.3) is not a matrix representation of $A_1^{(1)}$, merely a specification of its basis emphasizing its structure as a central extension of a loop algebra. The required product rules take the form:

$$\begin{aligned}&[h_j, h_k] = [e_j, e_k] = [f_j, f_k] = [h_j, \not{c}] = [e_k, \not{c}] = [f_k, \not{c}] = [\not{c}, \not{c}] = 0\\&[h_j, e_k] = 2e_{j+k} \qquad [h_j, f_k] = -2f_{j+k} \qquad [e_j, f_k] = h_{j+k} + j\delta_{j+k,0}\not{c}\end{aligned} \tag{3.4}$$

for all $j, k \in \mathbb{Z}$.

The algebra $A_1^{(1)}$ admits, amongst others, the following gradings:

$\mathbb{Z}_2$-gradings

$$\alpha : \begin{cases} L_0^\alpha = \{h_{2k}, e_{2k}, f_{2k}, k \in \mathbb{Z}, \quad \not{c}\} \\ L_1^\alpha = \{h_{2k+1}, e_{2k+1}, f_{2k+1}, k \in \mathbb{Z}\} \end{cases} \qquad \kappa^\alpha = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.5}$$

$$\beta : \begin{cases} L_0^\beta = \{h_k, k \in \mathbb{Z}, \quad \not{c}\} \\ L_1^\beta = \{e_k, f_k, k \in \mathbb{Z}\} \end{cases} \qquad \kappa^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.6}$$

$\mathbb{Z}_3$-gradings

$$\lambda : \begin{cases} L_0^\lambda = \{h_{3k}, e_{3k}, f_{3k} \quad k \in \mathbb{Z}, \quad \not{c}\} \\ L_1^\lambda = \{h_{3k+1}, e_{3k+1}, f_{3k+1} \quad k \in \mathbb{Z}\} \\ L_2^\lambda = \{h_{3k+2}, e_{3k+2}, f_{3k+2} \quad k \in \mathbb{Z}\} \end{cases} \qquad \kappa^\lambda = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.7}$$

$$\mu : \begin{cases} L_0^\mu = \{h_k \quad k \in \mathbb{Z}, \quad \not{c}\} \\ L_1^\mu = \{e_k \quad k \in \mathbb{Z}\} \\ L_2^\mu = \{f_k \quad k \in \mathbb{Z}\} \end{cases} \qquad \kappa^\mu = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{3.8}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$-grading

$$\eta : \begin{cases} L_{00}^\eta = \{h_{2k}, k \in \mathbb{Z}, \quad \not{c}\} \\ L_{01}^\eta = \{h_{2k+1}, k \in \mathbb{Z}\} \\ L_{10}^\eta = \{e_{2k}, f_{2k}, k \in \mathbb{Z}\} \\ L_{11}^\eta = \{e_{2k+1}, f_{2k+1}, k \in \mathbb{Z}\}. \end{cases} \qquad \kappa^\eta = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{3.9}$$

It is to be noted that the $\mathbb{Z}_2$-grading β, the $\mathbb{Z}_3$-grading μ and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading η are not generic, in the sense that the corresponding matrices κ contain some zeros. Whether or not the grading is generic, the grading subspace L_0 is a subalgebra of $A_1^{(1)}$. It so happens that the subalgebras L_0^β, L_0^μ and L_{00}^η are all Abelian. However, L_0^α, L_0^λ, $L_{00}^\eta + L_{10}^\eta$ and $L_{00}^\eta + L_{11}^\eta$ are all isomorphic to $A_1^{(1)}$. The complementary subspaces are representations of $A_1^{(1)}$ under the adjoint action. They are irreducible for the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings, and reducible as $L_1 \oplus L_2$ for the $\mathbb{Z}_3$-gradings.

4. Contractions of $A_1^{(1)}$

Once a grading of a Lie algebra is known one can determine all of its contractions which preserve that grading. In this section we do this for the $\mathbb{Z}_2$-, $\mathbb{Z}_3$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings of $A_1^{(1)}$ identified in the previous section. The contraction matrices ϵ, are found by solving (2.4). In the non-generic case it is important to remember that only a proper subset of (2.4*c*) applies, since the presence of factors $*$ renders some equations superfluous. In all cases there exist two trivial solutions to (2.4), namely the uncontracted solution $\epsilon = \kappa$ and the totally contracted (Abelian) solution $\epsilon = \mathbf{0}$. The full sets of solutions can be parametrized as follows.

The generic $\mathbb{Z}_2$-grading α. Contraction matrices ϵ^α:

$$\begin{pmatrix} x & x \\ x & y \end{pmatrix} \qquad \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

The non-generic $\mathbb{Z}_2$-grading β. Contraction matrices ϵ^β:

$$\begin{pmatrix} * & x \\ x & y \end{pmatrix}. \tag{4.2}$$

The generic $\mathbb{Z}_3$-grading λ. Contraction matrices ϵ^λ:

$$\begin{pmatrix} x & x & x \\ x & y & u \\ x & u & z \end{pmatrix} \qquad \text{with} \qquad xu = yz \tag{4.3a}$$

$$\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \qquad \text{with} \qquad yz = 0 \qquad \begin{pmatrix} x & x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}. \tag{4.3b}$$

The non-generic $\mathbb{Z}_3$-grading μ. Contraction matrices ϵ^μ:

$$\begin{pmatrix} * & x & x \\ x & * & z \\ x & z & * \end{pmatrix} \qquad \begin{pmatrix} * & x & y \\ x & * & 0 \\ y & 0 & * \end{pmatrix}. \tag{4.4}$$

The non-generic $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading η. Contraction matrices ϵ^η:

$$\begin{pmatrix} * & * & x & x \\ * & * & y & z \\ x & y & u & v \\ x & z & v & w \end{pmatrix} \qquad \text{with} \qquad ux = vy \quad uz = wy \quad vz = wx \tag{4.5a}$$

$$\begin{pmatrix} * & * & x & y \\ * & * & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & * & x & 0 \\ * & * & 0 & 0 \\ x & 0 & y & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} * & * & 0 & x \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 0 & y \end{pmatrix}. \tag{4.5b}$$

Each of the above contraction matrices may be renormalized by the independent scaling of all elements of each G-graded subspace $L_j{}^{\epsilon}$ of L^{ϵ} by a non-zero parameter a_j to give $L_j{}^{\epsilon'} = a_j L_j{}^{\epsilon}$. This has the effect of transforming ϵ to ϵ' with

$$\epsilon'_{jk} = \frac{a_j a_k}{a_{j+k}} \epsilon_{jk} \,. \tag{4.6}$$

The scaling matrix, ϵ^{s}, with elements

$$\epsilon^{\mathrm{s}}{}_{jk} = \frac{a_j a_k}{a_{j+k}} \tag{4.7}$$

is itself a contraction matrix satisfying (2.4). For the gradings under consideration here the scaling contraction matrices ϵ^{s} take the form

$$\mathbb{Z}_2 : \begin{pmatrix} a & a \\ a & b^2/a \end{pmatrix} \qquad \mathbb{Z}_3 : \begin{pmatrix} a & a & a \\ a & b^2/c & bc/a \\ a & bc/a & c^2/b \end{pmatrix}$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 : \begin{pmatrix} a & a & a & a \\ a & b^2/a & bc/d & bd/c \\ a & bc/d & c^2/a & cd/b \\ a & bd/c & cd/b & d^2/a \end{pmatrix} . \tag{4.8}$$

They may be used to renormalize each of the contraction matrices ϵ, given in (4.1)–(4.5) to one or other of the forms given in table 1. For example, ϵ^{η}, as given by (4.5a) with x, y, z, u, v, w all non-vanishing, may be renormalized by using ϵ^{s} as in (4.8) with

$$a = \frac{1}{x} \qquad b = \frac{1}{\sqrt{yz}} \qquad c = \frac{1}{\sqrt{xu}} = \frac{1}{\sqrt{yv}} \qquad d = \frac{1}{\sqrt{xw}} = \frac{1}{\sqrt{zv}} \,.$$

With these particular values we obtain

$$\epsilon^{\eta} \bullet \epsilon^{\mathrm{s}} = \begin{pmatrix} * & * & 1 & 1 \\ * & * & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

where $\bullet$ signifies the element by element product of matrices defined in (4.6).

It is notable that there are only two cases for which it is not possible to renormalize all the non-vanishing matrix elements to 1, namely

$$\epsilon^{\mu} = \begin{pmatrix} * & 1 & x \\ 1 & * & 0 \\ x & 0 & * \end{pmatrix} \qquad \text{and} \qquad \epsilon^{\eta} = \begin{pmatrix} * & * & 1 & x \\ * & * & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \end{pmatrix} \qquad \text{for} \quad 0 \neq x \neq 1 \,. \tag{4.9}$$

The contracted algebra L^{ϵ} may have quite a different structure from that of L. Indeed, this is what makes contractions interesting. To illustrate the types of structure that may emerge, consider the Lie algebra $L = A_1^{(1)}$ and its generic $\mathbb{Z}_2$-grading α defined in (3.5). As pointed out earlier, L_0^{α} is itself isomorphic to $A_1^{(1)}$.

Table 1. Renormalized contraction matrices ϵ for $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings. Here and in subsequent tables '$\cdot$' is used to indicate a zero matrix element, while $*$ is used to indicate an arbitrary matrix element.

Grading	ϵ	Independent solutions
$\mathbb{Z}_2 : \alpha$	$\begin{pmatrix} x & x \\ x & y \end{pmatrix}$	$(x, y) = (0, 0), (0, 1), (1, 0), (1, 1)$
	$\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}$	
$\mathbb{Z}_2 : \beta$	$\begin{pmatrix} * & x \\ x & y \end{pmatrix}$	$(x, y) = (0, 0), (0, 1), (1, 0), (1, 1)$
$\mathbb{Z}_3 : \lambda$	$\begin{pmatrix} x & x & x \\ x & y & u \\ x & u & z \end{pmatrix}$	$(x, y, z, u) = (0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 0), (0, 1, 0, 1), (1, 0, 0, 0), (1, 0, 1, 0), (1, 1, 0, 0), (1, 1, 1, 1)$
	$\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & y & \cdot \\ \cdot & \cdot & z \end{pmatrix}$	$(y, z) = (0, 0), (0, 1), (1, 0)$
	$\begin{pmatrix} 1 & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	
	$\begin{pmatrix} 1 & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$	
$\mathbb{Z}_3 : \mu$	$\begin{pmatrix} * & x & x \\ x & * & y \\ x & y & * \end{pmatrix}$	$(x, y) = (0, 0), (0, 1), (1, 0), (1, 1)$
	$\begin{pmatrix} * & x & y \\ x & * & \cdot \\ y & \cdot & * \end{pmatrix}$	$(x, y) = (0, 1), (1, 0), (1, r) \quad r \neq 0, 1$
$\mathbb{Z}_2 \times \mathbb{Z}_2 : \eta$	$\begin{pmatrix} * & * & x & x \\ * & * & y & z \\ x & y & u & v \\ x & z & v & w \end{pmatrix}$	$(x, y, z, u, v, w) = (0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 1, 1), (0, 0, 0, 1, 0, 0), (0, 0, 0, 1, 0, 1), (0, 0, 0, 1, 1, 0), (0, 0, 0, 1, 1, 1), (0, 0, 1, 0, 0, 0), (0, 0, 1, 0, 0, 1), (0, 1, 0, 0, 0, 0), (0, 1, 0, 1, 0, 0), (0, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 1), (1, 0, 0, 0, 0, 0), (1, 0, 0, 0, 1, 0), (1, 0, 1, 0, 0, 0), (1, 0, 1, 0, 1, 1), (1, 1, 0, 0, 0, 0), (1, 1, 0, 1, 1, 0), (1, 1, 1, 0, 0, 0), (1, 1, 1, 1, 1, 1)$
	$\begin{pmatrix} * & * & 1 & \cdot \\ * & * & \cdot & \cdot \\ 1 & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$x = 0, 1$
	$\begin{pmatrix} * & * & \cdot & 1 \\ * & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & x \end{pmatrix}$	$x = 0, 1$
	$\begin{pmatrix} * & * & 1 & x \\ * & * & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \end{pmatrix}$	$x \neq 0, 1$

In this case there exist three non-trivial contraction matrices, ϵ^α:

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{4.10}$$

For the first of these $[L_1^{\epsilon^\alpha}, L_1^{\epsilon^\alpha}] = 0$ so that $L_1^{\epsilon^\alpha}$ is Abelian, while $[L_0^{\epsilon^\alpha}, L_1^{\epsilon^\alpha}] = L_1^{\epsilon^\alpha}$. It follows that L^{ϵ^α} is an 'inhomogeneous' Kac–Moody algebra in the sense that it is isomorphic to $A_1^{(1)} + L_1^{\epsilon^\alpha}$, where $L_1^{\epsilon^\alpha}$ is an infinite Abelian algebra, playing the role of translations, which affords a representation of $L_0^{\epsilon^\alpha} = A_1^{(1)}$ under the adjoint action.

In contrast to this, for the second case of (4.10) $L_0^{\epsilon^\alpha}$ is still isomorphic to $A_1^{(1)}$ and the algebra $L^{\epsilon^\alpha} = A_1^{(1)} \oplus L_1^{\epsilon^\alpha}$ is decomposable. Once again $L_1^{\epsilon^\alpha}$ is an infinite Abelian algebra, but it now commutes with $L_0^{\epsilon^\alpha} = A_1^{(1)}$.

For the third case of (4.10) the contraction is such that $L_0^{\epsilon^\alpha}$ is now Abelian. The isomorphism with $A_1^{(1)}$ is lost. The full contracted algebra $L^{\epsilon^\alpha} = L_0^{\epsilon^\alpha} \oplus L_1^{\epsilon^\alpha}$ is again decomposable, but its subalgebras $L_0^{\epsilon^\alpha}$ and $L_1^{\epsilon^\alpha}$ are Abelian and nil-potent of degree 2, respectively.

In analysing the structure of the algebras obtained by the graded contraction of $A_1^{(1)}$ it is of interest to know the minimal set of vectors of the contracted algebra which generate the infinite set of all vectors corresponding to the positive root vectors of $A_1^{(1)}$. It is well known that the set of positive roots of $A_1^{(1)}$ is given by

$$\{p\alpha_0 + q\alpha_1 : p, q \in \mathbb{Z}^+, |p-q| \leqslant 1\}. \tag{4.11}$$

The corresponding root vectors of $A_1^{(1)}$, denoted by $E_{p\alpha_0+q\alpha_1}$, may be generated under the product $[\cdot,\cdot]$ by the simple root vectors E_{α_0} and E_{α_1}. The identification with the basis (3.3) is such that

$$\begin{aligned} E_{\alpha_0} &= \begin{pmatrix}0 & 0\\ t & 0\end{pmatrix} = f_1 \qquad & F_{\alpha_0} &= \begin{pmatrix}0 & t^{-1}\\ 0 & 0\end{pmatrix} = e_{-1}\\ E_{\alpha_1} &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = e_0 \qquad & F_{\alpha_1} &= \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = f_0 . \end{aligned} \tag{4.12}$$

Here $(E_{\alpha_0}, F_{\alpha_0}, H_{\alpha_0}) = (f_1, e_{-1}, \not{c} - h_0)$ and $(E_{\alpha_1}, F_{\alpha_1}, H_{\alpha_1}) = (e_0, f_0, h_0)$ span the three-dimensional simple Lie algebras A_1 corresponding to the simple roots α_0 and α_1, respectively, of $A_1^{(1)}$.

In general we have

$$\begin{aligned} E_{\alpha_0+k\delta} &= E_{(k+1)\alpha_0+k\alpha_1} = \begin{pmatrix}0 & 0\\ t^{k+1} & 0\end{pmatrix} = f_{k+1}\\ E_{\alpha_1+k\delta} &= E_{k\alpha_0+(k+1)\alpha_1} = \begin{pmatrix}0 & t^k\\ 0 & 0\end{pmatrix} = e_k\\ E_{k\delta} &= E_{k\alpha_0+k\alpha_1} = \begin{pmatrix}t^k & 0\\ 0 & -t^k\end{pmatrix} = h_k \end{aligned} \tag{4.13}$$

where $\delta = \alpha_0 + \alpha_1$ is a null-root of $A_1^{(1)}$.

Consider first the $\mathbb{Z}_2$-grading α defined in (3.5) and its contractions given in (4.4). In order to find the set of vectors generating all $E_{p\alpha_0+q\alpha_1}$ we need to distinguish between the families of $A_1^{(1)}$ root vectors in L_0^α and L_1^α. In this case we have

$$\begin{aligned} L_0^\alpha &= \{h_{2k}, e_{2k}, f_{2k}, k \in \mathbb{Z}; \not{c}\} = \{E_{2k\delta}, E_{\alpha_1+2k\delta}, E_{\alpha_0+(2k-1)\delta}, k \in \mathbb{Z}; \not{c}\}\\ L_1^\alpha &= \{h_{2k+1}, e_{2k+1}, f_{2k+1}, k \in \mathbb{Z}\} = \{E_{(2k+1)\delta}, E_{\alpha_1+(2k+1)\delta}, E_{\alpha_0+2k\delta}, k \in \mathbb{Z}\}. \end{aligned} \tag{4.14}$$

By way of example we now analyse the contraction defined by $\epsilon^\alpha = \binom{11}{10}$. For sufficiently large k, any of the vectors in (4.14) can be generated by products of the type $[L_0, L_0]$ or $[L_1, L_0]$ which, in contrast to those of type $[L_1, L_1]$, are not contracted to zero. Indeed we have

$$\left.\begin{aligned} E_{2k\delta} &= [E_{\alpha_1}, E_{\alpha_0+(2k-1)\delta}] \in [L_0, L_0] \\ E_{(2k-1)\delta} &= [E_{\alpha_1}, E_{\alpha_0+(2k-2)\delta}] \in [L_0, L_1] \\ 2E_{\alpha_0+2k\delta} &= [E_{\alpha_0}, E_{2k\delta}] \in [L_1, L_0] \\ 2E_{\alpha_0+(2k+1)\delta} &= [E_{\alpha_0+\delta}, E_{2k\delta}] \in [L_0, L_0] \\ -2E_{\alpha_1+2k\delta} &= [E_{\alpha_1}, E_{2k\delta}] \in [L_0, L_0] \\ -2E_{\alpha_1+(2k+1)\delta} &= [E_{\alpha_1+\delta}, E_{2k\delta}] \in [L_1, L_0] \end{aligned}\right\} \quad \text{for } k \geqslant 1 . \tag{4.15}$$

There remain just two non-simple positive root vectors of $A_1^{(1)}$ to consider, namely $E_{\alpha_0+\delta}$ and $E_{\alpha_1+\delta}$. For the latter

$$-2E_{\alpha_1+\delta} = [E_{\alpha_1}, E_\delta] \in [L_0, L_1] . \tag{4.16}$$

However, for the former

$$[E_{\alpha_0}, E_\delta] \in [L_1, L_1] \tag{4.17}$$

which vanishes under the given contraction. In fact $E_{\alpha_0+\delta} = E_{2\alpha_0+\alpha_1}$ cannot be generated from E_{α_0} and E_{α_1} under this contraction. It follows from these results, that in the case of the contraction defined by $\epsilon^\alpha = \binom{11}{10}$ vectors corresponding to all positive root vectors of $A_1^{(1)}$ are generated by

$$E_{\alpha_0} \qquad E_{\alpha_1} \qquad \text{and} \qquad E_{\alpha_0+\delta} . \tag{4.18}$$

The results of a similar analysis of all possible contractions of the two distinct Z_2-gradings, α and β, of $A_1^{(1)}$ given in (3.5) and (3.6) are displayed in table 2.

Table 2. Generators of the vectors corresponding to all positive root vectors of $A_1^{(1)}$ for all possible contractions of $\mathbb{Z}_2$-graded $A_1^{(1)}$.

Grading	Contraction	Generators
$\mathbb{Z}_2 : \alpha$	$\begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix}$	$E_{\alpha_0}, E_{\alpha_1}, E_{\alpha_0+\delta}$
	$\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}$	$E_{\alpha_0+\delta}, E_{\alpha_0+2k\delta}, E_{\alpha_1}, E_{\alpha_1+(2k+1)\delta}, E_{(2k+1)\delta}$ for $k \geqslant 0$
	$\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$	$E_{\alpha_0+2k\delta}, E_{\alpha_1}, E_{\alpha_1+(2k+1)\delta}, E_{(2k+1)\delta}$ for $k \geqslant 0$
$\mathbb{Z}_2 : \beta$	$\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}$	$E_{\alpha_0}, E_{\alpha_1}, E_{(k+1)\delta}$ for $k \geqslant 0$
	$\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$	$E_{\alpha_0+k\delta}, E_{\alpha_1+k\delta}$ for $k \geqslant 0$

5. Contractions of representations

Assuming that one has the Lie algebra L and its representations V graded by means of the same Abelian grading group G one can study the graded contractions of representations even without any detailed knowledge of the graded subspaces V_m of V. The details of the gradings of some representations of $A_1^{(1)}$ are discussed later in section 7. Here we merely assume that each representation has been graded in a manner consistent with the gradings of $A_1^{(1)}$ spelt out in section 3.

For each contraction specified by one or other of the matrices ϵ tabulated in section 4, (2.11) may be solved for all possible matrices ψ governing corresponding graded contractions of representations. In order to do this for a specific representation V it is necessary to know the matrix λ of L-module parameters (2.7). The graded representations of $A_1^{(1)}$ of interest here are all generic in the sense that $\lambda_{jm} = 1$ for all j and m. Henceforth this will be assumed to be the case, allowing the solutions ψ of (2.11) to be enumerated for each ϵ, independently of the particular representation under consideration.

It is to be noted that replacing m by $m+n$ for any n in (2.11b) leaves these equations unaltered except for a permutation of the columns of ψ determined by the action of the element n of the grading group G. This implies that if ψ is a solution of (2.11), then others may be obtained from ψ by suitable permutations of its columns.

Before embarking on the enumeration it is worth disposing of some scaling possibilities. All vectors in the G-graded subspace V_m of any L-module V may be independently scaled by a non-zero parameter b_m to give $V'_m = b_m V_m$ for each $m \in G$. This corresponds to changing ψ to $\psi' = \psi^s \bullet \psi$ with

$$\psi^s{}_{jm} = \frac{b_{j+m}}{b_m} \qquad \text{so that} \quad \psi'_{jm} = \frac{b_{j+m}}{b_m}\psi_{jm}\,. \tag{5.1}$$

If ψ satisfies (2.11) then so does ψ'. Such a scaling leaves not only all 0's unaltered but also the whole of the top row of ψ, as can be seen by setting $j = 0$ in (5.1). However the scaling allows any non-vanishing entry below the top row in the fixed column labelled by m to be renormalized to 1. The existence of 0's in this column will leave corresponding ratios of the scaling parameters free to renormalize entries in other columns.

Returning to the top row of ψ in the generic case for which $\kappa_{00} \neq 0$, it follows from (2.11b) that if $\epsilon_{00} = 0$ then all the entries in the top row of ψ are 0. Moreover, having renormalized any non-vanishing ϵ_{00} to 1, it also follows from (2.11b) that $\psi_{0m} = 0$ or 1 for all m. Hence in the generic case all the entries in the top row of ψ are 0 or 1. This is not true in the non-generic case.

The scaling matrices ψ^s are given for each of our grading groups by

$$\mathbb{Z}_2 : \begin{pmatrix} 1 & 1 \\ b/a & a/b \end{pmatrix} \qquad \mathbb{Z}_3 : \begin{pmatrix} 1 & 1 & 1 \\ b/a & c/b & a/c \\ c/a & a/b & b/c \end{pmatrix}$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 : \begin{pmatrix} 1 & 1 & 1 & 1 \\ b/a & a/b & d/c & c/d \\ c/a & d/b & a/c & b/d \\ d/a & c/b & b/c & a/d \end{pmatrix}. \tag{5.2}$$

All non-trivial solutions ψ of (2.11) are given, up to scaling and column permutations, in tables 3–7 for each renormalized contraction matrix ϵ of table 1. The scaling of a solution

ψ involving an arbitrary non-zero parameter p may be illustrated by the following typical example:

$$\psi^{\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ p & 0 & p & 0 \end{pmatrix} \qquad \psi^{s} \bullet \psi^{\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \tag{5.3}$$

where the required scaling matrix is obtained from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ matrix ψ^{s} given in (5.2) by setting $a = p$, $b = 1$, $c = p$, and $d = 1$. Other scalings are done in the same way.

Table 3. All possible non-trivial ψ (up to scaling and cyclic permutations of columns) for each $\mathbb{Z}_2$-graded renormalized contraction matrix ϵ^{α}.

ϵ	ψ
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$

Table 4. All possible non-trivial ψ (up to scaling and cyclic permutations of columns) for each renormalized $\mathbb{Z}_2$-graded contraction matrix ϵ^{β}. The parameters u and v are independent and arbitrary, and not subject to scaling.

ϵ	ψ
$\begin{pmatrix} * & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} * & 1 \\ 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} u & v \\ \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} * & \cdot \\ \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} u & v \\ \cdot & \cdot \end{pmatrix}$

It is notable that only in the non-generic cases are there solutions ψ containing non-vanishing entries which cannot be scaled to 1. These exceptional cases include, for example:

$$\psi^{\beta} = \begin{pmatrix} t & u \\ 0 & 0 \end{pmatrix} \qquad \psi^{\mu} = \begin{pmatrix} t & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \psi^{\mu} = \begin{pmatrix} 1 & 1 & 1 \\ p & q & 0 \\ r & 0 & 0 \end{pmatrix}$$
$$\psi^{\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & q & 0 & 0 \\ r & s & 0 & 0 \end{pmatrix}. \tag{5.4}$$

Of course, if any one of the parameters is 0 then no scaling of that parameter is required. Moreover, it may be shown that if any one or more of the unscaled parameters p, q, r, s

Table 5. All possible non-trivial ψ (up to scaling and cyclic permutations of columns) for each renormalized $\mathbb{Z}_3$-graded contraction matrix ϵ^λ.

ϵ	ψ
$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&1&\cdot\\1&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&1&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&1\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$ $\begin{pmatrix}1&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$ $\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&\cdot&1\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&1&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$ $\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&1&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\1&\cdot&1\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&1&1\\\cdot&1&\cdot\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\1&1&\cdot\\1&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&1\end{pmatrix}$	$\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\1&\cdot&1\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$
$\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&1&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}1&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$ $\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&\cdot\end{pmatrix}\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\end{pmatrix}$

Table 5. Continuned.

ϵ	ψ
$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}$

Table 6. All possible non-trivial ψ (up to scaling and cyclic permutations of columns) for each renormalized $\mathbb{Z}_3$-graded contraction matrix ϵ^{μ}. The parameters are subject to the constraints: (i) $x \in \{0, 1\}$; if $x = 0$ then $y \in \{0, 1\}$; if $x = 1$ then y is arbitrary; (ii) if $pqr = 0$ then p, q, $r \in \{0, 1\}$; if $pqr \neq 0$ then $p = q = 1$ with r arbitrary; (iii) u, v, w are independent and arbitrary.

ϵ	ψ
$\begin{pmatrix} * & 1 & 1 \\ 1 & * & 1 \\ 1 & 1 & * \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} * & 1 & 1 \\ 1 & * & \cdot \\ 1 & \cdot & * \end{pmatrix}$	$\begin{pmatrix} u & v & w \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & u \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & u \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & u \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 \\ p & q & r \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ p & q & \cdot \\ r & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ p & \cdot & \cdot \\ q & \cdot & r \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \cdot & \cdot & \cdot \\ p & q & r \end{pmatrix}$
$\begin{pmatrix} * & \cdot & \cdot \\ \cdot & * & 1 \\ \cdot & 1 & * \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ p & q & r \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ p & q & \cdot \\ r & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ p & \cdot & \cdot \\ q & \cdot & r \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p & q & r \end{pmatrix}$
$\begin{pmatrix} * & x & y \\ x & * & \cdot \\ y & \cdot & * \end{pmatrix}$	$\begin{pmatrix} u & v & w \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & u \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} y & y & u \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} x & x & u \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} y & y & u \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x \\ p & q & r \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} y & y & y \\ \cdot & \cdot & \cdot \\ p & q & r \end{pmatrix}$

appearing in rows of ψ other than the first is 0, then the remaining non-vanishing parameters may be rescaled to 1. Thus in the last two examples of (5.4), if $pqr \neq 0$ and $pqrs \neq 0$, respectively, then the solutions may be rescaled to be of the form:

$$\psi^{\mu} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ r & 0 & 0 \end{pmatrix} \qquad \psi^{\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & s & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Unfortunately, nothing can be done about parameters in the first row of ψ.

The compatibility between the contraction of the Lie algebra and of its representations allows us to determine representations of comparatively exotic algebras whose representation theory is little understood. For example, we identified the 'inhomogeneous' Kac–Moody algebra $A_1^{(1)} \Supset L_1^{\epsilon}$ by invoking the $\mathbb{Z}_2$-grading α of $A_1^{(1)}$ and the contraction specified by $\epsilon = \binom{11}{10}$. Non-trivial compatible contractions of representations of $A_1^{(1)}$ to give representations of $A_1^{(1)} \Supset L_1^{\epsilon}$ are then specified by $\psi = \binom{11}{10}$, $\binom{11}{00}$ and $\binom{10}{00}$.

Table 7. All possible non-trivial ψ (up to scaling and the permutations 1234, 2143, 3412, 4321 of columns) for each renormalized $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded contraction matrix ϵ^η. The parameters are subject to the constraints: (i) $x \in \{0, 1\}$; if $x = 0$ then $y = z = 1$; if $x = 1$ then $y \in \{0, 1\}$ and z is arbitrary; (ii) if $jk = 0$ then $j, k \in \{0, 1\}$; if $jk \neq 0$ then $j = 1$ with k arbitrary; (iii) if $mn = 0$ then $m, n \in \{0, 1\}$; if $mn \neq 0$ then $m = 1$ with n arbitrary; (iv) if $pqrs = 0$ then $p, q, r, s \in \{0, 1\}$; if $pqrs \neq 0$ then $p = q = r = 1$ with s arbitrary; (v) t, u, v, w are independent and arbitrary.

ϵ	ψ
$\begin{pmatrix} * & * & 1 & 1 \\ * & * & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} * & * & x & x \\ * & * & 1 & 1 \\ x & 1 & \cdot & \cdot \\ x & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} t & u & v & w \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & u \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & v \\ j & k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & 1 & 1 \\ * & * & 1 & \cdot \\ 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & 1 & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & 1 & 1 \\ * & * & \cdot & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}$
$\begin{pmatrix} * & * & 1 & \cdot \\ * & * & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & \cdot & 1 \\ * & * & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & \cdot & \cdot \\ * & * & 1 & 1 \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & \cdot & \cdot \\ * & * & \cdot & \cdot \\ \cdot & \cdot & x & 1 \\ \cdot & \cdot & 1 & y \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p & q & \cdot & \cdot \\ r & s & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & \cdot & \cdot \\ * & * & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & \cdot & \cdot \\ * & * & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$

Table 7. Continuted.

ϵ	ψ
$\begin{pmatrix} * & * & \cdot & \cdot \\ * & * & \cdot & \cdot \\ \cdot & \cdot & x & \cdot \\ \cdot & \cdot & \cdot & y \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ p & q & \cdot & \cdot \\ r & s & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & x & x \\ * & * & 1 & \cdot \\ x & 1 & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} t & u & v & w \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & u \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & v \\ j & k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & t & x & u \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & t \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & t & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 1 & \cdot & 1 & \cdot \\ 1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & x & x \\ * & * & \cdot & 1 \\ x & \cdot & \cdot & \cdot \\ x & 1 & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} t & u & v & w \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & u \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t & t & u & v \\ j & k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & t & u & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & t & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & t \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & x & x \\ * & * & \cdot & \cdot \\ x & \cdot & \cdot & 1 \\ x & \cdot & 1 & \cdot \end{pmatrix}$	$\begin{pmatrix} t & t & u & v \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & t & x & u \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & t & u & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & t & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & t \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ p & q & \cdot & \cdot \\ r & s & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ 1 & \cdot & 1 & \cdot \end{pmatrix} \begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \end{pmatrix}$

Consider the first of these with $\psi = \binom{11}{10}$. Under the $\mathbb{Z}_2$-grading α any representation $V^{\epsilon,\psi} = V_0^{\epsilon,\psi} + V_1^{\epsilon,\psi}$ is decomposable as a representation of $L_0^\epsilon = A_1^{(1)}$ since $L_0^\epsilon V_0^{\epsilon,\psi} \subseteq V_0^{\epsilon,\psi}$ and $L_0^\epsilon V_1^{\epsilon,\psi} \subseteq V_1^{\epsilon,\psi}$. However, the action of L_1^ϵ gives $L_1^\epsilon V_0^{\epsilon,\psi} \subseteq V_1^{\epsilon,\psi}$ and $L_1^\epsilon V_1^{\epsilon,\psi} = \emptyset$. Thus the 'translations' L_1^ϵ map $V_0^{\epsilon,\psi}$ to $V_1^{\epsilon,\psi}$ and annihilate $V_1^{\epsilon,\psi}$. It follows that under contraction each graded irreducible module $V = V_0 + V_1$ of $L = A_1^{(1)}$ gives a reducible, but indecomposable module $V^{\epsilon,\psi} = V_0^{\epsilon,\psi} \ni V_1^{\epsilon,\psi}$ of $L^\epsilon = A_1^{(1)} \ni L_1^\epsilon$.

On the other hand for $\psi = \binom{11}{00}$, L_1^ϵ annihilates both $V_0^{\epsilon,\psi}$ and $V_1^{\epsilon,\psi}$. Thus under

Table 7. Continued.

ϵ	ψ
$\begin{pmatrix} * & * & x & x \\ * & * & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} t & u & v & w \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} t & t & u & u \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} t & t & u & v \\ j & k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & t & x & u \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & t & u & x \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & t \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & t & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & t & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & t \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$ $\begin{pmatrix} x & t & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & t \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & x \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ p & q & \cdot & \cdot \\ r & s & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} * & * & x & z \\ * & * & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \\ z & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} t & u & v & w \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} t & t & u & u \\ j & k & m & n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} t & t & u & v \\ j & k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & t & x & u \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} x & x & x & t \\ 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & t & x \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ p & \cdot & \cdot & q \\ r & \cdot & \cdot & s \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} x & x & x & x \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
	$\begin{pmatrix} z & t & u & z \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} z & z & t & z \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ $\begin{pmatrix} z & z & z & t \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$ $\begin{pmatrix} z & z & z & z \\ p & \cdot & q & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r & \cdot & s & \cdot \end{pmatrix}$
	$\begin{pmatrix} z & z & z & z \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \end{pmatrix}$

contraction each graded irreducible module $V = V_0 + V_1$ of $L = A_1^{(1)}$ gives a decomposable module $V^{\epsilon,\psi} = V_0^{\epsilon,\psi} \oplus V_1^{\epsilon,\psi}$ of $L^\epsilon = A_1^{(1)} \ni L_1^\epsilon$. Since the action of L_1^ϵ is trivial, the passage from L to L^ϵ, together with the compatible contraction of representations specified by $\psi = \binom{11}{00}$ is akin to the restriction of L to its subalgebra L_0, together with the corresponding branching of representations. However, it should be emphasized that even if $V = V_0 + V_1$ is an irreducible representation of $L = A_1^{(1)}$ which is decomposable into $V_0^{\epsilon,\psi} \oplus V_1^{\epsilon,\psi}$ as a representation of L_0^ϵ, further reduction may be possible. This is taken up in section 8 where the grading of representations is made explicit and branching rules are calculated.

6. Contractions of tensor products of representations

Any tensor product of representations of a Lie algebra L is itself a representation (in general reducible) of L. However, after the contraction of L and its representations, the tensor product does not provide a representation of the contracted algebra L^ϵ unless the tensor product is also contracted. The requirements imposed on the tensor product contractions can again be described without a specific choice of representations.

For each contraction of representations specified by some particular matrix ψ given in section 5, (2.19) may be solved for all possible matrices τ governing corresponding contractions of tensor products of representations. For completeness it is also necessary to consider those matrices ψ obtained from those of section 5 by permutation of columns under the action of the grading group G. The linearity of (2.19) ensures that in each case the general solution τ may be expressed as an arbitrary linear combination of particular solutions. We denote the corresponding coefficients in such linear combinations by a, b, $c, \ldots$. These are independent parameters which may take on any value including zero.

Table 8. Row matrix constraints on tensor product contractions for $\mathbb{Z}_2$-gradings.

ρ	$\tau_0(\rho)$	$\tau_1(\rho)$
(1 1)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$
(1 0)	$\begin{pmatrix} a & \cdot \\ \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} a & a \\ a & \cdot \end{pmatrix}$
(0 1)	$\begin{pmatrix} a & \cdot \\ \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot \\ \cdot & d \end{pmatrix}$
(0 0)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Table 9. Row matrix constraints on tensor product contractions for $\mathbb{Z}_3$-gradings.

ρ	$\tau_0(\rho)$	$\tau_1(\rho)$	$\tau_2(\rho)$
(1 1 1)	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix}$	$\begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix}$
(1 1 0)	$\begin{pmatrix} a & b & \cdot \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} a & a & a \\ a & a & \cdot \\ a & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & e & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$
(1 0 1)	$\begin{pmatrix} a & \cdot & c \\ \cdot & \cdot & \cdot \\ g & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & i \end{pmatrix}$	$\begin{pmatrix} a & a & a \\ a & \cdot & \cdot \\ a & \cdot & a \end{pmatrix}$
(0 1 1)	$\begin{pmatrix} a & \cdot & \cdot \\ \cdot & e & \cdot \\ \cdot & \cdot & i \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & e & e \\ \cdot & e & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & f \\ \cdot & f & f \end{pmatrix}$
(1 0 0)	$\begin{pmatrix} a & \cdot & \cdot \\ \cdot & e & \cdot \\ \cdot & \cdot & i \end{pmatrix}$	$\begin{pmatrix} a & a & c \\ a & \cdot & \cdot \\ g & \cdot & i \end{pmatrix}$	$\begin{pmatrix} a & b & a \\ d & e & \cdot \\ a & \cdot & \cdot \end{pmatrix}$
(0 1 0)	$\begin{pmatrix} a & \cdot & c \\ \cdot & \cdot & \cdot \\ g & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} a & \cdot & \cdot \\ \cdot & e & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & e & f \\ \cdot & h & \cdot \end{pmatrix}$
(0 0 1)	$\begin{pmatrix} a & b & \cdot \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & f \\ \cdot & h & i \end{pmatrix}$	$\begin{pmatrix} a & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & i \end{pmatrix}$
(0 0 0)	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

In solving (2.19) it is helpful to note that these equations may be subdivided into disjoint sets of equations. Each set corresponds to a distinct value of j and depends only upon the elements in the jth row of the matrix ψ. It is therefore convenient to introduce a row matrix ρ, defined to coincide with the jth row of ψ for some fixed $j \in G$, so that

$$\rho_m = \psi_{jm} \qquad \text{for all} \quad m \in G\,. \tag{6.1}$$

The relevant set of equations (2.19) may then be rewritten in the form

$$\rho_{m+n}\tau_j(\rho)_{mn} = \rho_m\tau_j(\rho)_{j+m,n} = \rho_n\tau_j(\rho)_{m,j+n} \tag{6.2}$$

where $\tau_j(\rho)$ is used to denote the general solution, τ, to (2.19) corresponding to the fixed j and ρ in question. To recover the general solution τ to (2.19) for any particular ψ it is then only necessary to impose the constraints:

$$\tau = \tau_j(\rho) \qquad \text{with} \quad \rho_m = \psi_{jm} \quad \text{for all} \quad j, m \in G\,. \tag{6.3}$$

Table 10. Tensor product contractions for $\mathbb{Z}_2$-gradings.

τ	ψ
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} a & a \\ a & a \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} a & a \\ a & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$
$\begin{pmatrix} a & \cdot \\ \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$
$\begin{pmatrix} \cdot & \cdot \\ \cdot & d \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ \cdot & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$

For the grading groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ the general solutions, $\tau_j(\rho)$, are displayed in a suitably parametrized form in tables 8 and 9, respectively, for each possible pair j and ρ, with entries of ρ restricted, on the grounds of economy, to the values 0 and 1. In the case of $\mathbb{Z}_2$-gradings for each ψ the resulting matrices τ are displayed in table 10. In the case of $\mathbb{Z}_3$-gradings we content ourselves with a single illustration of the method of calculation based on (6.1)–(6.3). In the case

$$\psi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.4}$$

we have

$$\tau = \tau_0(1\,1\,1) = \tau_1(1\,0\,0) = \tau_2(1\,0\,0)\,. \tag{6.5}$$

It then follows from table 9 that

$$\tau = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & a & c \\ a & 0 & 0 \\ g & 0 & i \end{pmatrix} = \begin{pmatrix} a & b & a \\ d & e & 0 \\ a & 0 & 0 \end{pmatrix} \tag{6.6}$$

so that $a = b = c = d = g$ and $e = f = h = i = 0$. Hence, for ψ as given in (6.4), the general solution τ to (2.19) takes the form

$$\tau = \begin{pmatrix} a & a & a \\ a & 0 & 0 \\ a & 0 & 0 \end{pmatrix}. \tag{6.7}$$

More complete results for $\mathbb{Z}_3$-gradings may be found elsewhere [12]. Those for $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings are too numerous to display.

7. Gradings of representations of $A_1^{(1)}$

The gradings of a particular highest-weight representation of $A_1^{(1)}$ are described explicitly for each of the gradings described in section 3. The method applies equally well to any other representation which allows a weight-space decomposition of the representation space.

Each irreducible highest-weight integrable module V^λ of $A_1^{(1)}$ is labelled by its highest-weight $\lambda = \lambda_0\omega_0 + \lambda_1\omega_1 - k\delta = (\lambda_0, \lambda_1; k)$ with $\lambda_0, \lambda_1, k \in \mathbb{Z}^+$. The level, L, and depth, d, of λ are given by $L = \lambda_0 + \lambda_1$ and $d = k$, respectively. Such a module has a weight-space decomposition $V^\lambda = \oplus_{\sigma\in H^*} V_\sigma^\lambda$, where the dimension of V_σ^λ is the weight multiplicity m_σ^λ of the weight $\sigma = \sigma_0\omega_0 + \sigma_1\omega_1 - d\delta = (\sigma_0, \sigma_1; d)$.

Each of the gradings of $A_1^{(1)}$ defined in section 3 leads, as in (2.5), to a corresponding grading of each $A_1^{(1)}$-module V^λ. To illustrate the nature of these gradings it is useful to consider the weight-space decomposition of some highest-weight irreducible representation. By way of example the weight-space decomposition of the module $V^{(1,0;0)}$ down to depth $d = 10$ takes the form [14]

(σ_0, σ_1)	$\cdots(7,-6)$	$(5,-4)$	$(3,-2)$	$(1,0)$	$(-1,2)$	$(-3,4)$	$(-5,6)\cdots$
$d=0$				1			
$d=1$			1	1	1		
$d=2$			1	2	1		
$d=3$			2	3	2		
$d=4$		1	3	5	3	1	
$d=5$		1	5	7	5	1	
$d=6$		2	7	11	7	2	
$d=7$		3	11	15	11	3	
$d=8$		5	15	22	15	5	
$d=9$	1	7	22	30	22	7	1
$d=10$	1	11	30	42	30	11	1

where the weight multiplicities appearing in any vertical string are the numbers of partitions, $p(n)$, of n generated by $\prod_{k=1}^{\infty}(1-q^k)^{-1} = \sum_{n=0}^{\infty} p(n)q^n$.

Each of the gradings of section 3 leads to a corresponding grading $V^\lambda = \oplus_m V_m^\lambda$ which is consistent with the weight-space decomposition in the sense that, for all weights σ with $m_\sigma^\lambda \neq 0$, $V_\sigma^\lambda \subset V_m^\lambda$ for some m. Each point in the weight-space can therefore be unambiguously assigned a grading m. This has been carried out below for the module $V^{(1,0;0)}$ for each of the gradings of section 3.

$\mathbb{Z}_2$-grading α:

(σ_0, σ_1)	$\cdots(7,-6)$	$(5,-4)$	$(3,-2)$	$(1,0)$	$(-1,2)$	$(-3,4)$	$(-5,6)\cdots$
$d=0$				0			
$d=1$			1	1	1		
$d=2$			0	0	0		
$d=3$			1	1	1		
$d=4$		0	0	0	0	0	
$d=5$		1	1	1	1	1	
$d=6$		0	0	0	0	0	
$d=7$		1	1	1	1	1	
$d=8$		0	0	0	0	0	
$d=9$	1	1	1	1	1	1	1
$d=10$	0	0	0	0	0	0	0

$\mathbb{Z}_2$-grading β:

(σ_0, σ_1)	$\cdots$ (7, −6)	(5, −4)	(3, −2)	(1, 0)	(−1, 2)	(−3, 4)	(−5, 6) $\cdots$
$d=0$				0			
$d=1$			1	0	1		
$d=2$			1	0	1		
$d=3$			1	0	1		
$d=4$		0	1	0	1	0	
$d=5$		0	1	0	1	0	
$d=6$		0	1	0	1	0	
$d=7$		0	1	0	1	0	
$d=8$		0	1	0	1	0	
$d=9$	1	0	1	0	1	0	1
$d=10$	1	0	1	0	1	0	1

$\mathbb{Z}_3$-grading λ:

(σ_0, σ_1)	$\cdots$ (7, −6)	(5, −4)	(3, −2)	(1, 0)	(−1, 2)	(−3, 4)	(−5, 6) $\cdots$
$d=0$				0			
$d=1$			2	2	2		
$d=2$			1	1	1		
$d=3$			0	0	0		
$d=4$		2	2	2	2	2	
$d=5$		1	1	1	1	1	
$d=6$		0	0	0	0	0	
$d=7$		2	2	2	2	2	
$d=8$		1	1	1	1	1	
$d=9$	0	0	0	0	0	0	0
$d=10$	2	2	2	2	2	2	2

$\mathbb{Z}_3$-grading μ:

(σ_0, σ_1)	$\cdots$ (7, −6)	(5, −4)	(3, −2)	(1, 0)	(−1, 2)	(−3, 4)	(−5, 6) $\cdots$
$d=0$				0			
$d=1$			1	0	2		
$d=2$			1	0	2		
$d=3$			1	0	2		
$d=4$		2	1	0	2	1	
$d=5$		2	1	0	2	1	
$d=6$		2	1	0	2	1	
$d=7$		2	1	0	2	1	
$d=8$		2	1	0	2	1	
$d=9$	0	2	1	0	2	1	0
$d=10$	0	2	1	0	2	1	0

$\mathbb{Z}_2 \times \mathbb{Z}_2$-grading η:

(σ_0, σ_1)	$\cdots (7,-6)$	$(5,-4)$	$(3,-2)$	$(1,0)$	$(-1,2)$	$(-3,4)$	$(-5,6) \cdots$
$d=0$				00			
$d=1$			11	01	11		
$d=2$			10	00	10		
$d=3$			11	01	11		
$d=4$		00	10	00	10	00	
$d=5$		01	11	01	11	01	
$d=6$		00	10	00	10	00	
$d=7$		01	11	01	11	01	
$d=8$		00	10	00	10	00	
$d=9$	11	01	11	01	11	01	11
$d=10$	10	00	10	00	10	00	10

Combining the results displayed in the weight-multiplicity diagram with each of the above grading lattices enables each graded subspace $V_m^{(1,0;0)}$ to be identified.

It is not necessary to assign the grading $m = 0$ to the highest weight $(1, 0; 0)$ of the irreducible module $V^{(1,0;0)}$, as has been done in each of the above displays. In general the highest weight may be assigned any grading value and each graded subspace $V_m^{(1,0;0)}$ is found by positioning the weight-multiplicity diagram on the periodic grading lattice in just such a way that the highest weight falls at an appropriately labelled point.

It is a trivial matter to apply the same gradings to any other module V^λ. It is merely necessary to assign the highest weight some particular grading and to position the relevant weight-multiplicity diagram so that the highest weight falls at a point in the grading lattice labelled by this value of the grading. Each grading subspace V_m^λ is then identified by the intersection of the weight-multiplicity diagram of V^λ with the sublattice of the grading lattice consisting of those points labelled by m.

By virtue of (2.5), which includes the special case $L_0 V_m \subseteq V_m$, each graded subspace $V_m^{(1,0;0)}$ serves to define an L_0-module. In some cases it is easy to identify these modules. In particular, this can be done in those cases for which L_0 is isomorphic to $L = A_1^{(1)}$. This is true, as pointed out in section 3, for the $\mathbb{Z}_2$-grading α, and the $\mathbb{Z}_3$-grading λ. We shall return to this in section 8.

8. Branching rules defined by gradings

It follows as special cases of (2.1) and (2.5) that $[L_0, L_0] \subseteq L_0$ and $L_0 V_m \subseteq V_m$ for all $m \in G$, respectively. Hence each graded subspace V_m serves to define an L_0-module. In particular, in those cases we have identified for which $L = L_0 = A_1^{(1)}$ a consideration of the graded subspaces illustrated in section 6 enables the projection between weight spaces of $A_1^{(1)}$ in the reduction $A_1^{(1)} \supset A_1^{(1)}$ to be carried out explicitly, thereby leading to an evaluation of the corresponding branching rule.

This procedure may be illustrated as follows for the case of the $\mathbb{Z}_2$-grading α. Consider first the irreducible representation $V^{(1,0;0)} = V_0^{(1,0;0)} + V_1^{(1,0;0)}$. The graded decomposition of the weight space takes the form

$d=0$				1							0			
$d=1$			0	0	0					1	1	1		
$d=2$			1	2	1					0	0	0		
$d=3$			0	0	0					2	3	2		
$d=4$		1	3	5	3	1			0	0	0	0	0	
$d=5$		0	0	0	0	0			1	5	7	5	1	
$d=6$		2	7	11	7	2			0	0	0	0	0	
$d=7$		0	0	0	0	0			3	11	15	11	3	
$d=8$		5	15	22	15	5			0	0	0	0	0	
$d=9$	0	0	0	0	0	0	0	1	7	22	30	22	7	1
$d=10$	1	11	30	42	30	11	1	0	0	0	0	0	0	0.

Changing the scale on which the depth is measured, each of the above two sets of level-1 weights $(\sigma_0, \sigma_1; d)$ of $A_1^{(1)}$, along with their multiplicities, may be associated with level-2 weights $(\tau_0, \tau_1; e)$ of the isomorphic subalgebra $A_1^{(1)}$ to give for $V_0^{(1,0;0)}$

(τ_0, τ_1)	$\cdots$	$(8,-6)$	$(6,-4)$	$(4,-2)$	$(2,0)$	$(0,2)$	$(-2,4)$	$(-4,6)$	$\cdots$
$e=0$					1				
$e=1$				1	2	1			
$e=2$			1	3	5	3	1		
$e=3$			2	7	11	7	2		
$e=4$			5	15	22	15	5		
$e=5$		1	11	30	42	30	11	1	

and for $V_1^{(1,0;0)}$

(τ_0, τ_1)	$\cdots$	$(8,-6)$	$(6,-4)$	$(4,-2)$	$(2,0)$	$(0,2)$	$(-2,4)$	$(-4,6)$	$\cdots$
$e=1/2$				1	1	1			
$e=3/2$				2	3	2			
$e=5/2$			1	5	7	5	1		
$e=7/2$			3	11	15	11	3		
$e=9/2$		1	7	22	30	22	7	1.	

The weight-space decomposition of the irreducible level 2 module $V^{(2,0;0)}$ is

(τ_0, τ_1)	$\cdots$	$(8,-6)$	$(6,-4)$	$(4,-2)$	$(2,0)$	$(0,2)$	$(-2,4)$	$(-4,6)$	$\cdots$
$e=0$					1				
$e=1$				1	1	1			
$e=2$			1	2	3	2	1		
$e=3$			2	4	5	4	1		
$e=4$			3	7	10	7	1		
$e=5$		1	5	13	16	13	5	1	

and that of $V^{(0,2;0)}$:

(τ_0, τ_1)	$\cdots$	$(8,-6)$	$(6,-4)$	$(4,-2)$	$(2,0)$	$(0,2)$	$(-2,4)$	$(-4,6)$	$\cdots$
$e=0$				1	1	1			
$e=1$				1	2	1			
$e=2$			1	3	4	3	1		
$e=3$			2	5	7	5	2		
$e=4$		1	4	10	13	10	4	1.	

These may be exploited to determine the first few terms of the required branching rules. The procedure is to systematically subtract from the weights of $V_0^{(1,0;0)}$ and $V_1^{(1,0;0)}$ various copies of the weights of $V^{(2,0;e)}$ and $V^{(0,2;e)}$ along with those branching-rule multiplicities necessary to give zero at successive depths $e = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

The above data gives

$$\begin{aligned} V_0^{(1,0;0)} &= V^{(2,0;0)} + V^{(2,0;1)} + V^{(2,0;2)} + 2V^{(2,0;3)} + 2V^{(2,0;4)} + 3V^{(2,0;5)} + \cdots \\ V_1^{(1,0;0)} &= V^{(0,2;1/2)} + V^{(0,2;3/2)} + V^{(0,2;5/2)} + 2V^{(0,2;7/2)} + 2V^{(0,2;9/2)} + \cdots . \end{aligned} \tag{8.1}$$

It is not necessary to examine any weights other than those in the dominant sector since all the others are related to these by appropriate affine Weyl-group reflections. Working with previously tabulated strings of weight multiplicities it is easy to extend the above results.

The multiplicities of the weights $(1, 0; d)$ in $V^{(1,0;0)}$ take the form

$$m_{(1,0;d)}^{(1,0;0)} = 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, \ldots \tag{8.2}$$

for $d = 0, 1, 2, \ldots$, so that those of $V_0^{(1,0;0)}$ and $V_1^{(1,0;0)}$ are given by

$$\begin{aligned} m_{(1,0;e)}^{(1,0;0)_0} &= 1, 2, 5, 11, 22, 42, 77, 135, 231, 385, 627, 1002, \ldots \\ m_{(1,0;e)}^{(1,0;0)_1} &= 1, 3, 7, 15, 30, 56, 101, 176, 297, 490, 792, 1255, \ldots \end{aligned} \tag{8.3}$$

for $e = d/2 = 0, 1, 2, \ldots$ and $\frac{1}{2}, \frac{3}{2}, \ldots$, respectively. Similarly the weight multiplicities of $(2, 0; e)$ and $(0, 2; e)$ in both $V^{(2,0;0)}$ and $V^{(0,2;0)}$ are given by:

$$\begin{aligned} m_{(2,0;e)}^{(2,0;0)} &= 1, 1, 3, 5, 10, 16, 28, 43, 70, 105, 161, 236, \ldots \\ m_{(0,2;e)}^{(2,0;0)} &= 0, 1, 2, 4, 7, 13, 21, 35, 55, 86, 130, 196, \ldots \\ m_{(2,0;e)}^{(0,2;0)} &= 1, 2, 4, 7, 13, 21, 35, 55, 86, 130, 196, 287, 420, \ldots \\ m_{(0,2;e)}^{(0,2;0)} &= 1, 1, 3, 5, 10, 16, 28, 43, 70, 105, 161, 236, 350, \ldots \end{aligned} \tag{8.4}$$

for $e = 0, 1, 2, \ldots$.

Denoting the branching multiplicity of V^{μ} in V_m^{λ} by $b_{\mu}^{\lambda_m}$, it follows from (8.3) and (8.4) that

$$\begin{aligned} b_{(2,0;e)}^{(1,0;0)_0} &= 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, \ldots \\ b_{(0,2;e)}^{(1,0;0)_0} &= 0 \end{aligned} \tag{8.5}$$

for $e = 0, 1, 2, \ldots$, and

$$\begin{aligned} b_{(2,0;e)}^{(1,0;0)_1} &= 0 \\ b_{(0,2;e)}^{(1,0;0)_1} &= 1, 1, 1, 2, 2, 3, 4, 5, 6, 8, 10, 12, \ldots \end{aligned} \tag{8.6}$$

for $e = \frac{1}{2}, \frac{3}{2}, \ldots$.

These results may be generalized as follows through the use of generating functions. It is well known that the string function $c_{(1,0)}^{(1,0)}$ which serves to generate as coefficients of powers of q the weight multiplicities $m_{(1,0;d)}^{(1,0;0)}$ of $V^{(1,0;0)} = V_0^{(1,0;0)} + V_1^{(1,0;0)}$ coincides with the generating function for $p(d)$, the number of partitions of d, that is

$$c_{(1,0)}^{(1,0)} = c_{(1,0)}^{(1,0;0)_0} + c_{(1,0)}^{(1,0;0)_1} = \prod_{k=1}^{\infty}(1-q^k)^{-1} = \sum_d m_{(1,0;d)}^{(1,0;0)}\, q^d\,. \tag{8.7}$$

Taking note of the fact that for the $\mathbb{Z}_2$-grading α the grading, m, of each weight is 0 or 1 according to whether its depth, d, is even or odd, respectively, it follows that the corresponding generating function for the weight multiplicities, or string function, of $V_0^{(1,0;0)} - V_1^{(1,0;0)}$ is just

$$\begin{aligned} c_{(1,0)}^{(1,0;0)_0} - c_{(1,0)}^{(1,0;0)_1} &= \prod_{k=1}^{\infty}(1-(-q)^k)^{-1} = \prod_{k=1}^{\infty}(1-q^{2k})^{-1}(1+q^{2k-1})^{-1} \\ &= \prod_{k=1}^{\infty}(1+q^{2k})(1-q^{2k})^{-1}(1+q^k)^{-1}\,. \end{aligned} \tag{8.8}$$

However, the string function for $V^{(2,0;0)} - V^{(0,2;1/2)}$ is given by

$$c_{(2,0)}^{(2,0;0)} - c_{(2,0)}^{(0,2;1/2)} = \prod_{k=1}^{\infty}(1-q^{k/2})(1-q^k)^{-2} = \prod_{k=1}^{\infty}(1-q^k)^{-1}(1+q^{k/2})^{-1}\,. \tag{8.9}$$

Making due allowance for the change of scale in the direction of δ in passing from weights of level 1 to weights of level 2, it follows from (8.8) and (8.9) that

$$c_{(1,0)}^{(1,0;0)_0} - c_{(1,0)}^{(1,0;0)_1} = \prod_{k=1}^{\infty}(1+q^k)\cdot\left(c_{(2,0)}^{(2,0;0)} - c_{(2,0)}^{(0,2;1/2)}\right). \tag{8.10}$$

Similarly

$$c_{(-1,2)}^{(1,0;0)_0} - c_{(-1,2)}^{(1,0;0)_1} = \prod_{k=1}^{\infty}(1+q^k)\cdot\left(c_{(0,2)}^{(2,0;0)} - c_{(0,2)}^{(0,2;1/2)}\right). \tag{8.11}$$

The generating function for the branching multiplicities is therefore given by

$$\begin{aligned} &b_{(2,0)}^{(1,0;0)_0} = \prod_{k=1}^{\infty}(1+q^k) \qquad b_{(0,2)}^{(1,0;0)_0} = 0 \\ &b_{(2,0)}^{(1,0;0)_1} = 0 \qquad b_{(0,2)}^{(1,0;0)_1} = q^{1/2}\prod_{k=1}^{\infty}(1+q^k)\,. \end{aligned} \tag{8.12}$$

This confirms and generalizes (8.5) and (8.6).

The technique used here in the case of the particular irreducible representation $V^{(1,0)}$ may be readily applied to others. The key point is that in passing from the weights $(\sigma_0, \sigma_1; d)$ of $A_1^{(1)}$ to corresponding weights $(\tau_0, \tau_1; e)$ of the isomorphic subalgebra $A_1^{(1)}$ the relevant mapping is defined by

$$(\sigma_0, \sigma_1; d) \quad\to\quad (\tau_0, \tau_1; e) = (2\sigma_0 + \sigma_1, \sigma_1; \tfrac{1}{2}d)\,. \tag{8.13}$$

Table 11. Branching multiplicities $b^{\lambda_m}_{\mu} = b^{(\lambda_0,\lambda_1;0)_m}_{(\mu_0,\mu_1;\epsilon)}$ associated with the decomposition $V^{\lambda}_m = \oplus_\mu b^{\lambda_m}_{\mu} V^{\mu}$ arising from the restriction of $L = A^{(1)}_1$ to its subalgebra $L_0 = A^{(1)}_1$ defined by the $\mathbb{Z}_2$-grading α given in (3.5).

$\mathbb{Z}_2$-grading α $(\lambda_0,\lambda_1;0)_m$	$\epsilon-(m/2)$ $(\mu_0,\mu_1;\epsilon)$	0	1	2	3	4	5
$(1,0;0)_0$	$(2,0;\epsilon)$	1	1	1	2	2	3
$(1,0;0)_1$	$(0,2;\epsilon)$	1	1	1	2	2	3
$(0,1;0)_0$	$(1,1;\epsilon)$	1		1	1	2	2
$(0,1;0)_1$	$(1,1;\epsilon)$	1	1	1	1	2	2
$(2,0;0)_0$	$(4,0;\epsilon)$	1	1	2	3	5	7
	$(2,2;\epsilon)$			1	1	3	4
	$(0,4;\epsilon)$		1	1	2	3	5
$(2,0;0)_1$	$(4,0;\epsilon)$					1	1
	$(2,2;\epsilon)$	1	2	3	5	8	12
	$(0,4;\epsilon)$						1
$(1,1;0)_0$	$(3,1;\epsilon)$	1	1	2	4	6	10
	$(1,3;\epsilon)$		1	2	3	5	8
$(1,1;0)_1$	$(3,1;\epsilon)$	1	2	3	5	8	12
	$(1,3;\epsilon)$	1	1	2	4	6	10
$(0,2;0)_0$	$(4,0;\epsilon)$			1	1	2	3
	$(2,2;\epsilon)$	1	1	2	4	6	9
	$(0,4;\epsilon)$				1	1	2
$(0,2;0)_1$	$(4,0;\epsilon)$	1	1	2	3	5	7
	$(2,2;\epsilon)$	1	1	2	3	5	8
	$(0,4;\epsilon)$		1	1	2	3	5
$(3,0;0)_0$	$(6,0;\epsilon)$	1	1	2	4	6	10
	$(4,2;\epsilon)$			1	2	4	8
	$(2,4;\epsilon)$		1	2	4	7	12
	$(0,6;\epsilon)$						
$(3,0;0)_1$	$(6,0;\epsilon)$					1	2
	$(4,2;\epsilon)$	1	2	4	7	12	20
	$(2,4;\epsilon)$			1	2	4	8
	$(0,6;\epsilon)$		1	1	2	4	6
$(2,1;0)_0$	$(5,1;\epsilon)$	1	1	3	5	10	16
	$(3,3;\epsilon)$		1	3	6	11	19
	$(1,5;\epsilon)$		1	1	3	5	10
$(2,1;0)_1$	$(5,1;\epsilon)$	1	2	4	7	13	21
	$(3,3;\epsilon)$	1	2	4	8	14	25
	$(1,5;\epsilon)$		1	2	4	7	13
$(1,2;0)_0$	$(6,0;\epsilon)$			1	2	4	7
	$(4,2;\epsilon)$	1	2	4	8	14	24
	$(2,4;\epsilon)$		1	2	4	8	14
	$(0,6;\epsilon)$			1	2	3	6
$(1,2;0)_1$	$(6,0;\epsilon)$	1	2	3	6	10	16
	$(4,2;\epsilon)$	1	2	4	8	14	24
	$(2,4;\epsilon)$	1	2	4	8	14	24
	$(0,6;\epsilon)$				1	2	4
$(0,3;0)_0$	$(5,1;\epsilon)$		1	2	4	7	12
	$(3,3;\epsilon)$	1	1	2	5	8	14
	$(1,5;\epsilon)$			1	2	4	7
$(0,3;0)_1$	$(5,1;\epsilon)$	1	1	3	5	9	15
	$(3,3;\epsilon)$	1	2	3	6	11	18
	$(0,5;\epsilon)$		1	1	3	5	9

Table 12. Branching multiplicities $b_\mu^{\lambda_m} = b_{(\mu_0,\mu_1;\epsilon)}^{(\lambda_0,\lambda_1;0)_m}$ associated with the decomposition $V_m^\lambda = \oplus_\mu b_\mu^{\lambda_m} V^\mu$ arising from various restrictions of $L = A_1^{(1)}$ to its subalgebras $L_0 = A_1^{(1)}$ defined by the $\mathbb{Z}_2$-grading α and the $\mathbb{Z}_3$-gradings λ.

$\mathbb{Z}_2$-grading α $(\lambda_0,\lambda_1;0)_m$	$\epsilon = \epsilon_0+\epsilon_1$ $(\mu_0,\mu_1;\epsilon)$	ϵ_1 ϵ_0	0	1	2	3	4	5
$(1,0;0)_0$	$(2,0;\epsilon)$	0	1	1	1	2	2	3
$(1,0;0)_1$	$(0,2;\epsilon)$	$\frac{1}{2}$	1	1	1	2	2	3
$(0,1;0)_0$	$(1,1;\epsilon)$	0	1		1	1	2	2
$(0,1;0)_1$	$(1,1;\epsilon)$	$\frac{1}{2}$	1	1	1	1	2	2
$\mathbb{Z}_3$-grading λ $(\lambda_0,\lambda_1;0)_m$	$\epsilon = \epsilon_0+\epsilon_1$ $(\mu_0,\mu_1;\epsilon)$	ϵ_1 ϵ_0	0	1	2	3	4	5
$(1,0;0)_0$	$(3,0;\epsilon)$	0	1	1	2	3	6	9
	$(1,2;\epsilon)$	0		1	2	4	7	12
$(1,0;0)_1$	$(3,0;\epsilon)$	$\frac{2}{3}$	1	1	3	4	8	12
	$(1,2;\epsilon)$	$\frac{2}{3}$	1	2	3	6	10	17
$(1,0;0)_2$	$(3,0;\epsilon)$	$\frac{1}{3}$		1	2	4	6	11
	$(1,2;\epsilon)$	$\frac{1}{3}$	1	1	3	5	9	14
$(0,1;0)_0$	$(2,1;\epsilon)$	0	1	1	3	5	9	14
	$(0,3;\epsilon)$	0			1	2	4	6
$(0,1;0)_1$	$(2,1;\epsilon)$	$\frac{2}{3}$	1	2	4	7	12	19
	$(0,3;\epsilon)$	$\frac{2}{3}$	1	1	2	3	6	9
$(0,1;0)_2$	$(2,1;\epsilon)$	$\frac{1}{3}$	1	2	3	6	10	17
	$(0,3;\epsilon)$	$\frac{1}{3}$		1	1	3	4	8

The corresponding branching multiplicities are displayed in table 11 for all V^λ with level $L = \lambda_0 + \lambda_1 \leqslant 3$ down to depth $e < 6$.

Turning to alternative gradings, as pointed out at the end of section 7, the $\mathbb{Z}_2$-grading α and the $\mathbb{Z}_3$-grading λ are both such that ${A_1^{(1)}}_0$ is isomorphic to $A_1^{(1)}$. The gradings of the representations V^λ and the branching of V_m^λ in accordance with the formula

$$V_m^\lambda = \oplus_\mu b_\mu^{\lambda_m} V^\mu$$

which may be determined as before. The relevant mappings take the form

$$(\sigma_0,\sigma_1;d) \to (\tau_0,\tau_1;e) = \begin{cases} \left(2\sigma_0+\sigma_1,\sigma_1;\tfrac{1}{2}d\right) & \mathbb{Z}_2\text{-grading } \alpha \\ \\ \left(3\sigma_0+2\sigma_1,\sigma_1;\tfrac{1}{3}d\right) & \mathbb{Z}_3\text{-grading } \lambda \end{cases} \tag{8.14}$$

and the corresponding branching multiplicities for level-one representations are given to depth less than 6 in table 12. These branching rules, and indeed the corresponding embeddings of $A_1^{(1)}$ in $A_1^{(1)}$, do not coincide with the branching rule, and the corresponding embedding, used elsewhere [13] to illustrate the notion of subjoinings within a Kac–Moody algebra context.

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